Physics 541 **April 21, 2010**

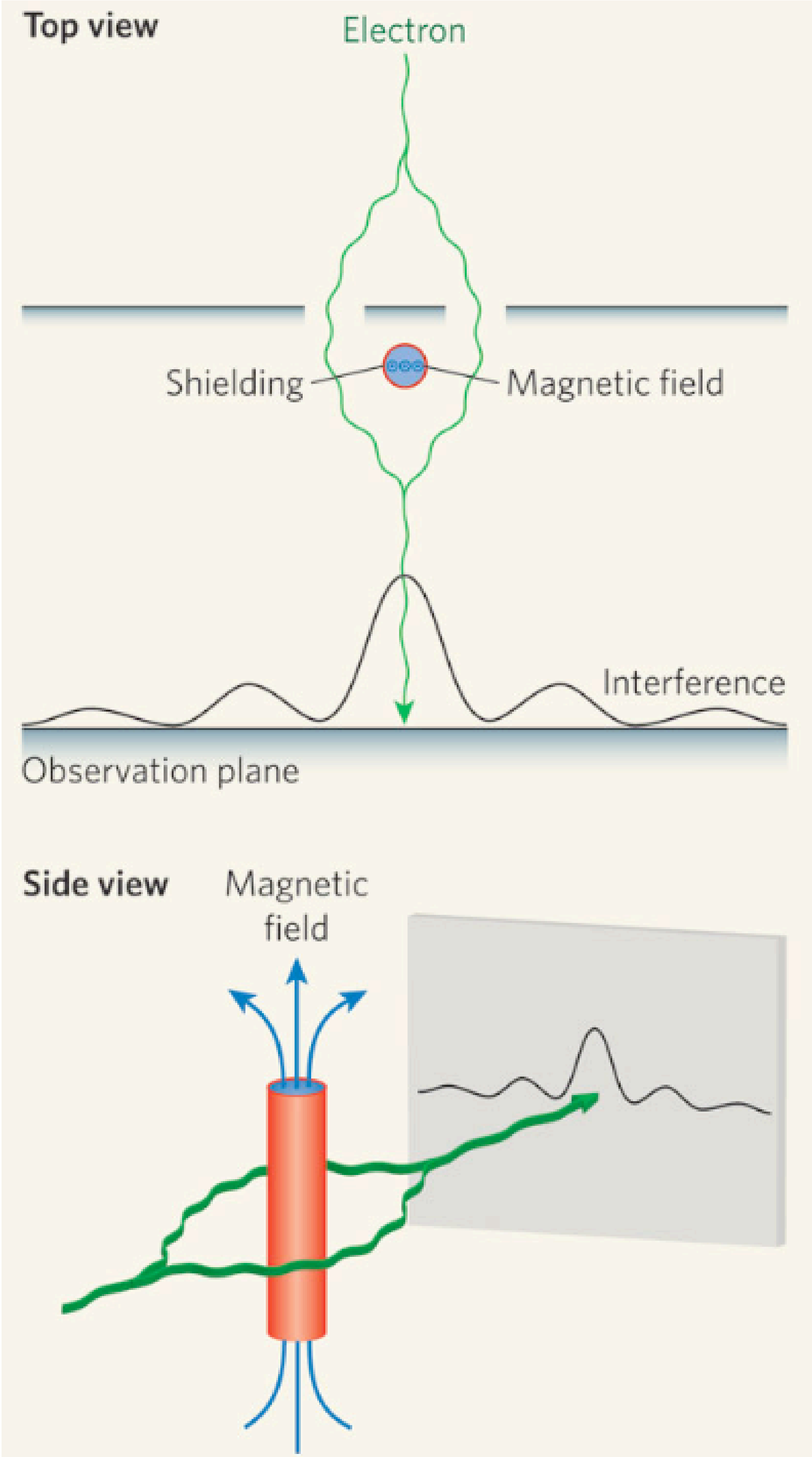
The Aharonov-Bohm Effect:

Topology
The Vector Potential
and
Gauge Transformations

The Aharonov-Bohm Effect

An electron moving in a region where E and B are zero, but A is not exhibits physical effects.

Therefore A is real whereas E and B are not.


Top view
Electron
Shielding
Magnetic field
Interference
Observation plane
Side view
Magnetic
field

The Vector Potential

EM Field Carries Momentum

Vector potential A => Photons

Gauge Transformations

ϕ determines E

and

A determines B

But ϕ and A are not unique

Gauge Transformations produce the different choices of ϕ and A that give the same E and B

Lots of Gauges

Coulomb Gauge

Lorenz Gauge

Axial Gauge

Temporal Gauge

Velocity Gauge

Kirchhoff Gauge

Landau Gauge

Feynman Gauge

t' Hooft Gauge

Unitary Gauge

Feynman's Paradox

A paradox is a situation which gives one answer when analyzed one way, and a different answer when analyzed another way, so that we are left in somewhat of a quandary as to actually what would happen. Of course, in physics there are never any real paradoxes because there is one correct answer; at least we believe that nature will act in only one way (and that is the *right way*, naturally). So a paradox in physics is only a confusion in our understanding.

act in only one way (and that is the *right way*, naturally). So in physics a paradox is only a confusion in our own understanding. Here is our paradox.

Imagine that we construct a device like that shown in Fig. 17–5. There is a thin, circular plastic disc supported on a concentric shaft with excellent bearings, so that it is quite free to rotate. On the disc is a coil of wire in the form of a short solenoid concentric with the axis of rotation. This solenoid carries a steady current I provided by a small battery, also mounted on the disc. Near the edge of the disc and spaced uniformly around its circumference are a number of small metal spheres insulated from each other and from the solenoid by the plastic material of the disc. Each of these small conducting spheres is charged with the same electrostatic charge Q. Everything is quite stationary, and the disc is at rest. Suppose now that by some accident—or by prearrangement—the current in the solenoid is interrupted, without, however, any intervention from the outside. So long as the current continued, there was a magnetic flux through the solenoid more or less parallel to the axis of the disc. When the current is interrupted, this flux must go to zero. There will, therefore, be an electric field induced which will circulate around in circles centered at the axis. The charged spheres on the perimeter of the disc will all experience an electric field tangential to the perimeter of the disc. This electric force is in the same sense for all the charges and so will result in a net torque on the disc. From these arguments we would expect that as the current in the solenoid disappears, the disc would begin to rotate. If we knew the moment of inertia of the disc, the current in the solenoid, and the charges on the small spheres, we could compute the resulting angular velocity.

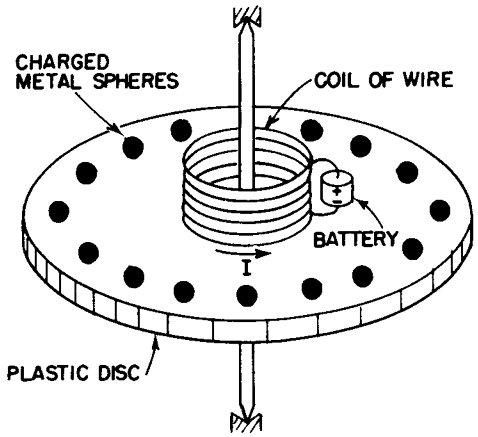


Fig. 17–5. Will the disc rotate if the current I is stopped?

But we could also make a different argument. Using the principle of the conservation of angular momentum, we could say that the angular momentum of the disc with all its equipment is initially zero, and so the angular momentum of the assembly should remain zero. There should be no rotation when the current is stopped. Which argument is correct? Will the disc rotate or will it not? We will leave this question for you to think about.

We should warn you that the correct answer does not depend on any nonessential feature, such as the asymmetric position of a battery, for example. In fact, you can imagine an ideal situation such as the following: The solenoid is made of superconducting wire through which there is a current. After the disc has been carefully placed at rest, the temperature of the solenoid is allowed to rise slowly. When the temperature of the wire reaches the transition temperature between superconductivity and normal conductivity, the current in the solenoid will be brought to zero by the resistance of the wire. The flux will, as before, fall to zero, and there will be an electric field around the axis. We should also warn you that the solution is not easy, nor is it a trick. When you figure it out, you will have discovered an important principle of electromagnetism.

17–5 Alternating-current generator

In the remainder of this chapter we apply the principles of Section 17–1 to analyze a number of the phenomena discussed in Chapter 16. We first look in more detail at the alternating-current generator. Such a generator consists basically of a coil of wire rotating in a uniform magnetic field. The same result can also be achieved by a fixed coil in a magnetic field whose direction rotates in the manner described in the last chapter. We will consider only the former case. Suppose we have a circular coil of wire which can be turned on an axis along one of its diameters. Let this coil be located in a uniform magnetic field perpendicular to the axis of rotation, as in Fig. 17–6. We also imagine that the two ends of the coil are brought to external connections through some kind of sliding contacts.

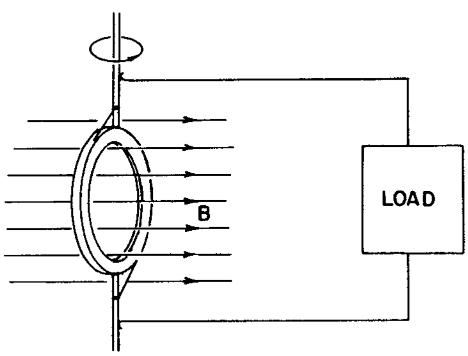


Fig. 17–6. A coil of wire rotating in a uniform magnetic field—the basic idea of the ac generator.

Due to the rotation of the coil, the magnetic flux through it will be changing. The circuit of the coil will therefore have an emf in it. Let S be the area of the coil and θ the angle between the magnetic field and the normal to the plane of the coil.*

* Now that we are using the letter A for the vector potential, we prefer to let S stand for a Surface area.

Generalized Momentum π

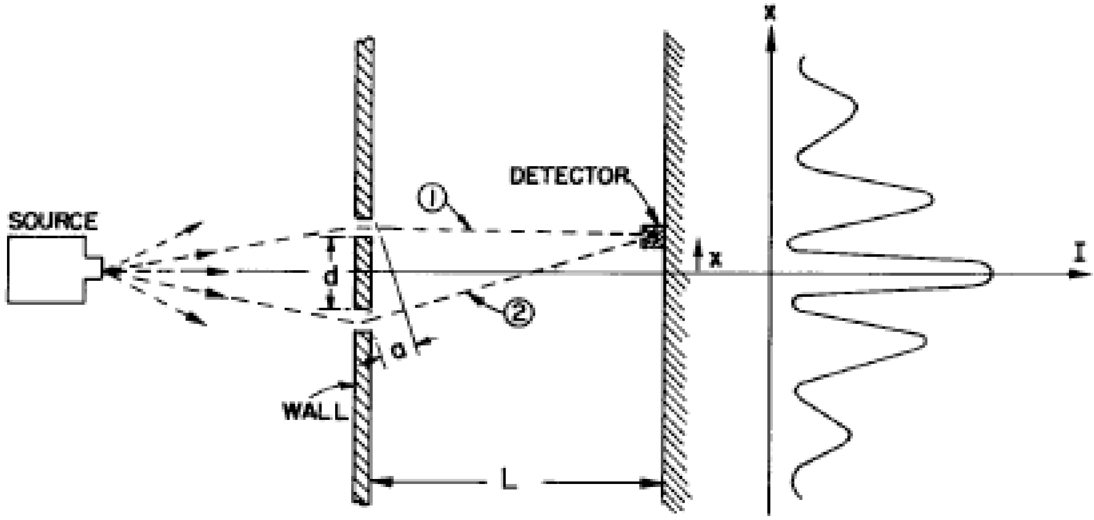
Particle Momentum p

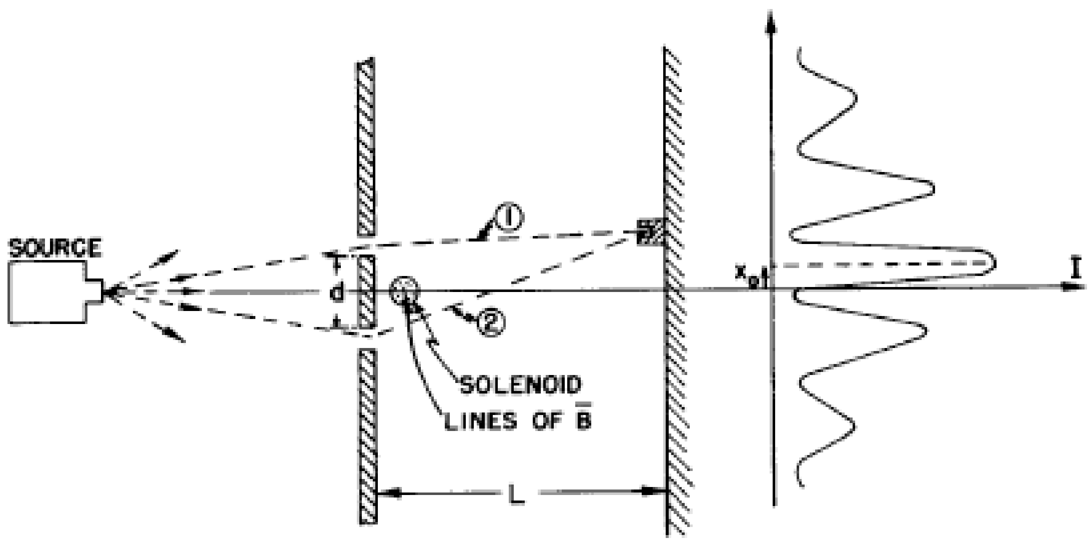
Field Momentum (e/c)A

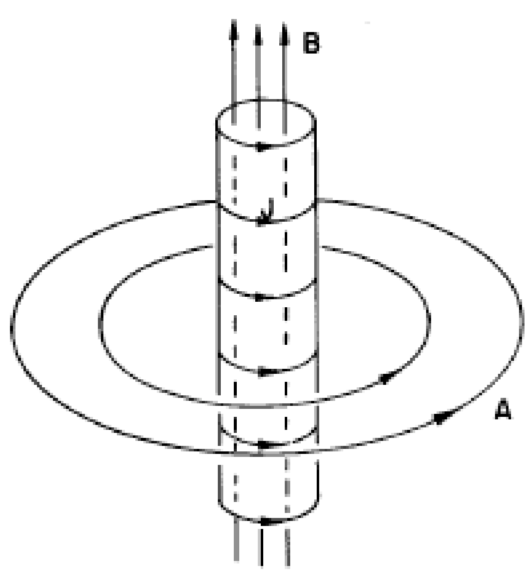
$$\pi = p - (e/c)\,A$$

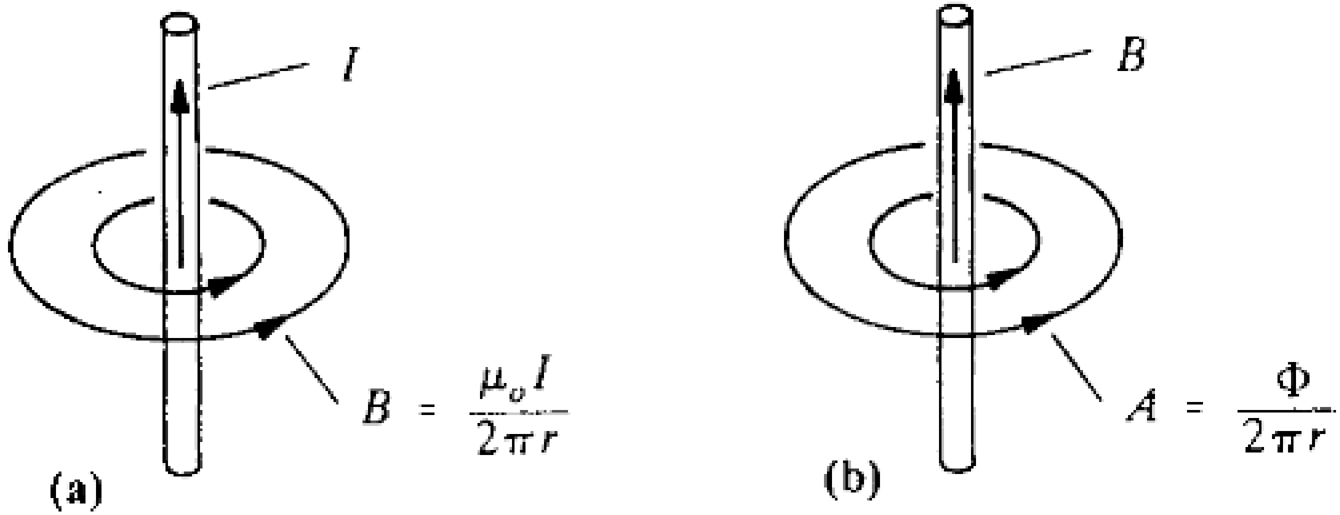
$$KE = \pi^2 / 2m$$

$$\pi^2 = p^2 - 2\,(e/c)\,p \cdot A + (e/c)^2\,A^2$$

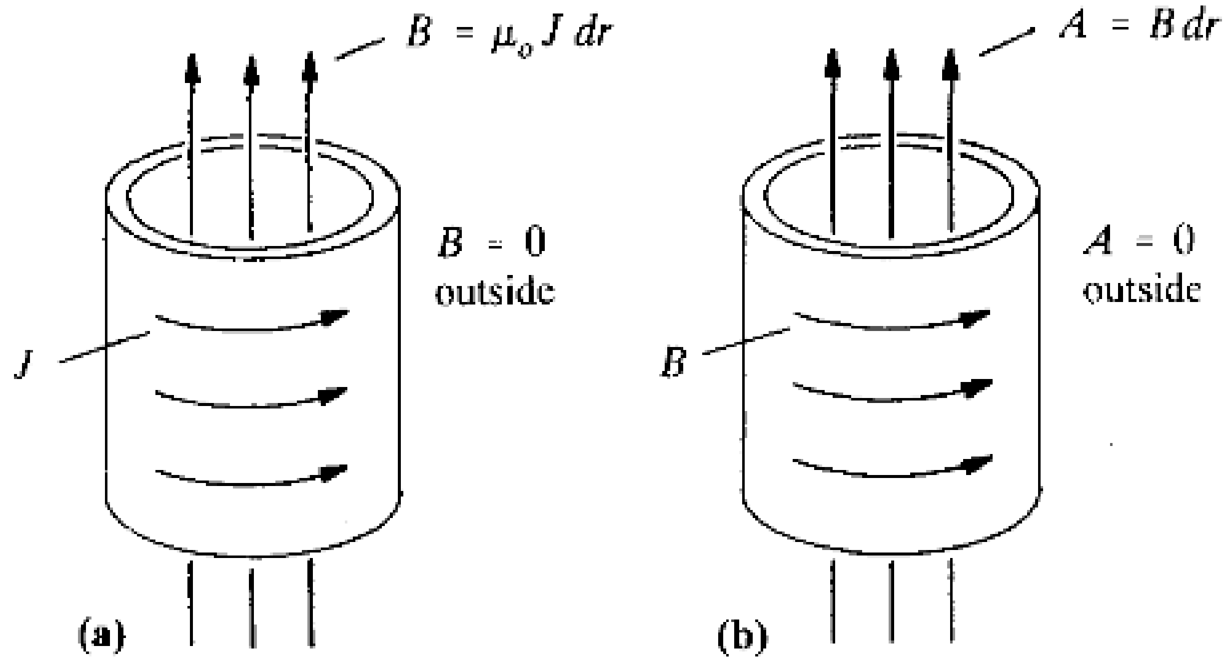

x
DETECTOR
SOURCE
1
x
d
I
2
a
WALL
L


1
SOURCE
d
2
SOLENOID
LINES OF B̄
I
L


B
A



I
$B = \frac{\mu_o I}{2\pi r}$
(a)
B
$A = \frac{\Phi}{2\pi r}$
(b)



$B = \mu_o J\,dr$
$B = 0$ outside
J
(a)
$A = B\,dr$
$A = 0$ outside
B
(b)

Gauge transformations

Electric and magnetic fields can be written in terms of scalar and vector p

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t},$$

$$\mathbf{B} = \nabla\times\mathbf{A}.$$

However, this prescription is not unique. There are many different potenti problem before. It is called *gauge invariance*. The most general transforn and (386) is

$$\phi \rightarrow \phi + \frac{\partial \psi}{\partial t},$$

$$\mathbf{A} \rightarrow \mathbf{A} - \nabla\psi.$$

This is clearly a generalization of the gauge transformation which we fou

$$\phi \rightarrow \phi + c,$$

$$\mathbf{A} \rightarrow \mathbf{A} - \nabla\psi,$$

where c is a constant. In fact, if $\psi(\mathbf{r},t) \rightarrow \psi(\mathbf{r}) + c\,t$ then Eqs. (387) ar

We are free to choose the gauge so as to make our equations as simple as potential is to make it go to zero at infinity:

$$\phi(\mathbf{r}) \rightarrow 0 \qquad \text{as} \quad |$$

$$
\begin{aligned}
\psi(\vec{r},t) &\rightarrow e^{i\lambda(\vec{r},t)}\psi(\vec{r},t) \\
\vec{A} &\rightarrow \vec{A} - \vec{\nabla} f(\vec{r},t) \\
\phi &\rightarrow \phi + \frac{1}{c}\frac{\partial f(\vec{r},t)}{\partial t} \\
f(\vec{r},t) &= \frac{\hbar c}{e}\lambda(\vec{r},t).
\end{aligned}
$$

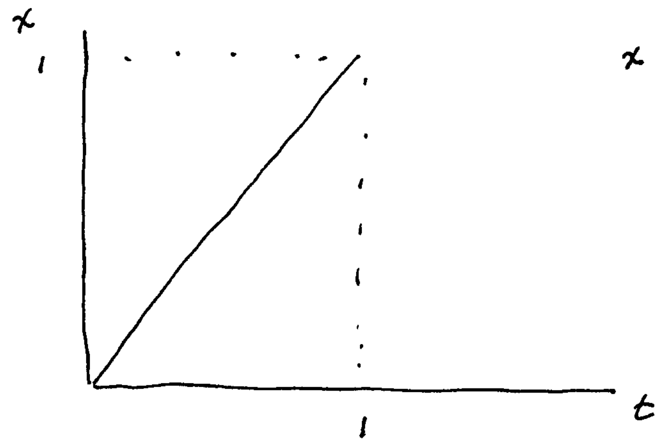
$\Delta S \leq \pi \hbar$

Q: Why do baseballs follow classical trajectories while electrons diffract, go thru both slits, etc.?

LAGRANGIAN for baseball and electron

$$L = T - V = \frac{1}{2} m v^2 - V(x) = \frac{1}{2} m v^2$$

THE CLASSICAL PATH



$x = v t \Rightarrow x = t$

CALCULATE THE ACTION FOR THE CLASSICAL PATH

$$S_{CL} = \int_0^1 L \, dt$$

$$= \int_0^1 \frac{1}{2} m v^2 \, dt$$

$$= \int_0^1 \frac{1}{2} m (1)^2 \, dt$$

$$S_{CL} = \frac{1}{2} m$$

NOW LET'S DO A NON CLASSICAL PATH

$$x = t^2$$

$$v = \frac{dx}{dt} = 2t$$

$$S_{NC} = \int_0^1 L \, dt = \int_0^1 \frac{1}{2} m (2t)^2 \, dt = \frac{4}{3}\left(\frac{1}{2} m\right)$$

$$S_{NC} = \frac{4}{3} S_{CL}$$ 14

THE DIFFRERENCE IS THE MASS!

BASEBALL $m \sim 200g$

$$S_{CL} = \frac{1}{2}(200g)(1\ cm/s)^2(1s)$$

$$= 100\ erg \cdot sec \approx 10^{29}\hbar$$

$$\Delta S = \frac{1}{3} S_{CL} = 3 \times 10^{28}\hbar \ggg \pi\hbar$$

SO, BASEBALL MUST STAY EXTREMELY CLOSE TO THE CLASSICAL PATH!

ELECTRON $m \sim 10^{-27} g$

$$S_{CL} = \frac{1}{2}(10^{-27})(1)^2(1)$$

$$= 5 \times 10^{-28}\ erg \cdot sec$$

~~$\frac{1}{2}\hbar < \pi\hbar$~~

$$\Delta S = \frac{1}{3} S_{CL} = \frac{1}{6}\hbar \ll \pi\hbar$$

SO, ELECTRON WILL FOLLOW THE NON CLASSICAL PATH!

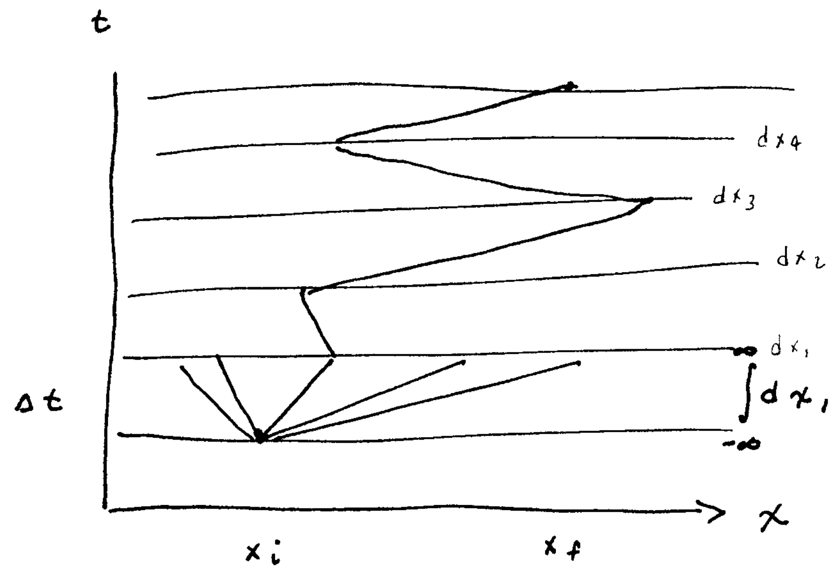
HOW TO DO THE PATH INTEGRAL?

$$A \sum_{\text{all paths}} e^{i S[x(t)]/\hbar}$$

$$A \int_{x_i}^{x_f} e^{i S[x(t)]/\hbar} \, D[x(t)]$$

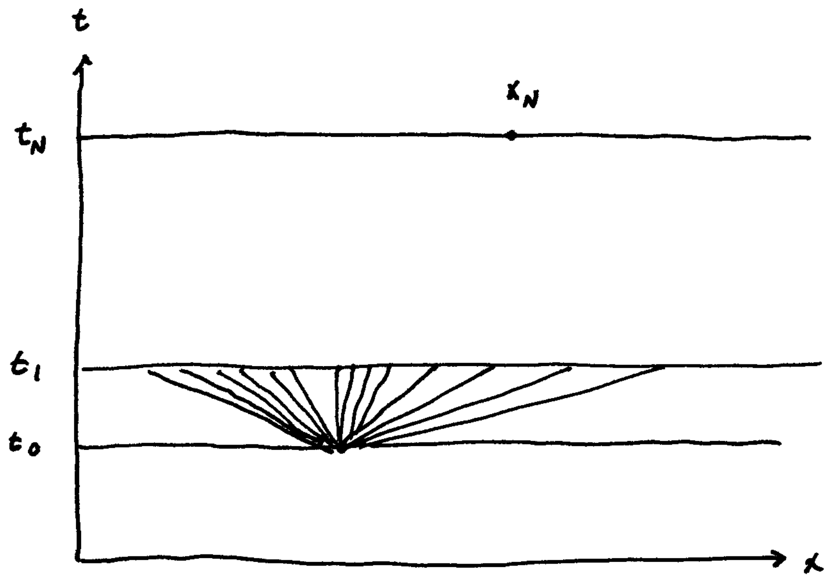
FUNCTIONAL INTEGRAL

$S[x(t)]$ IS A FUNCTIONAL



$$\lim_{\Delta t \to 0}$$

TIME SLICING



GO TO EVERY x_1

CONTINUOUS ACTION

$$S = \int_{t_0}^{t_N} L(t)\, dt = \int_{t_0}^{t_N} \frac{1}{2} m v^2 \, dt$$

DISCRETE APPROXIMATION $\qquad x_i = x(t_i)$

$$S = \sum_{i=0}^{N-1} \frac{1}{2} m \left(\frac{x_{i+1} - x_i}{\epsilon} \right)^2 \epsilon$$

PATH INTEGRAL

$$U(x_N, t_N; x_0, t_0) = \int_{x_0}^{x_N} e^{i S[x(t)]/\hbar} \, D[x(t)]$$

$$= \lim_{\substack{N \to \infty \\ \epsilon \to 0}} A \int_{-\infty}^{\infty} \int \cdots \int e^{\frac{i}{\hbar} \frac{m}{2} \sum_{i=0}^{N} (x_{i+1} - x_i)^2 / \epsilon} \, dx_1 \cdots dx_{N-1}$$

BECAUSE THE INTEGRAND IS GAUSSIAN,

WE CAN DO THE INTEGRALS!

DO THE dx_1 INTEGRATION

$$\int_{-\infty}^{\infty} \exp\frac{?}{?}\Big[\overset{y_2 - y_1}{(x_2 - x_1)^2} + \overset{y_1 - y_2}{(x_1 - x_0)^2} \Big]^{/(1)i} dx_1 \sim e^{-(y_2 - y_0)^2/2i}$$

DO THE dx_2 INTEGRATION

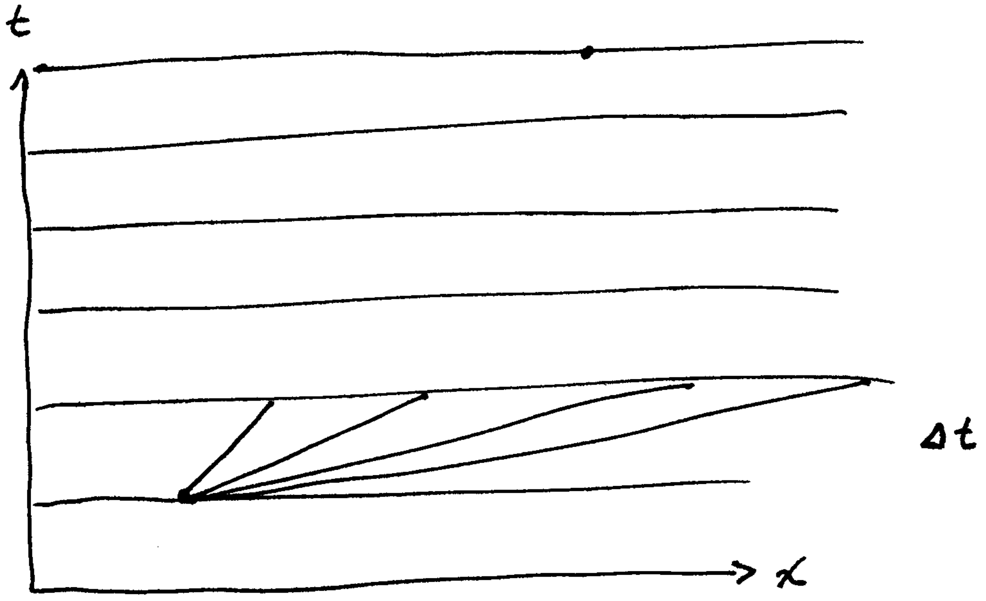
$$\int \qquad dx_2 \sim e^{-(y_3 - y_0)^2/3i}$$

$\vdots$

$$\sim e^{-(y_N - y_0)^2/Ni}$$

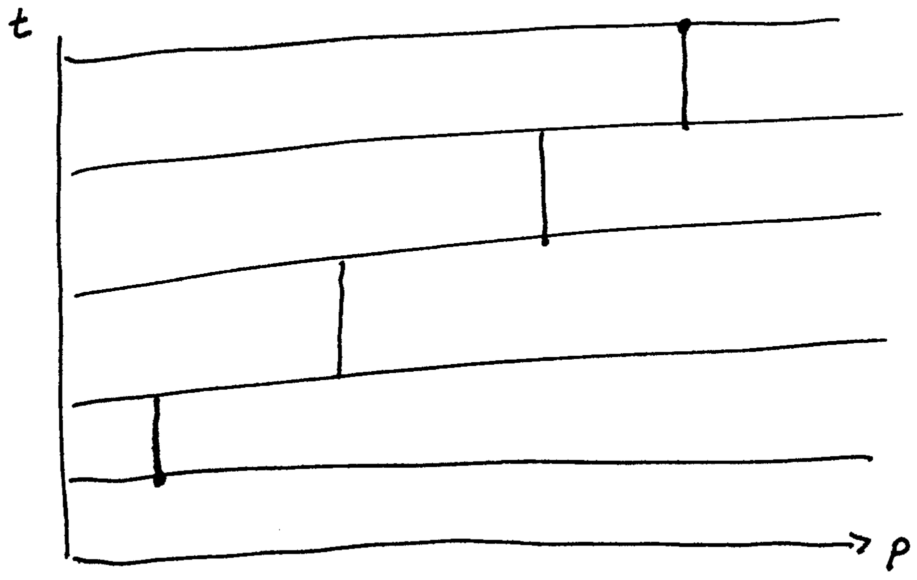
$$\exp\left[\frac{im\,(x_N - x_0)^2}{2\hbar N\epsilon}\right] \qquad \begin{array}{l} N \to \infty \\ \epsilon \to 0 \\ N\epsilon \to t_N - t_0 \end{array}$$

NRQM ⟹ INFINITE VELOCITIES

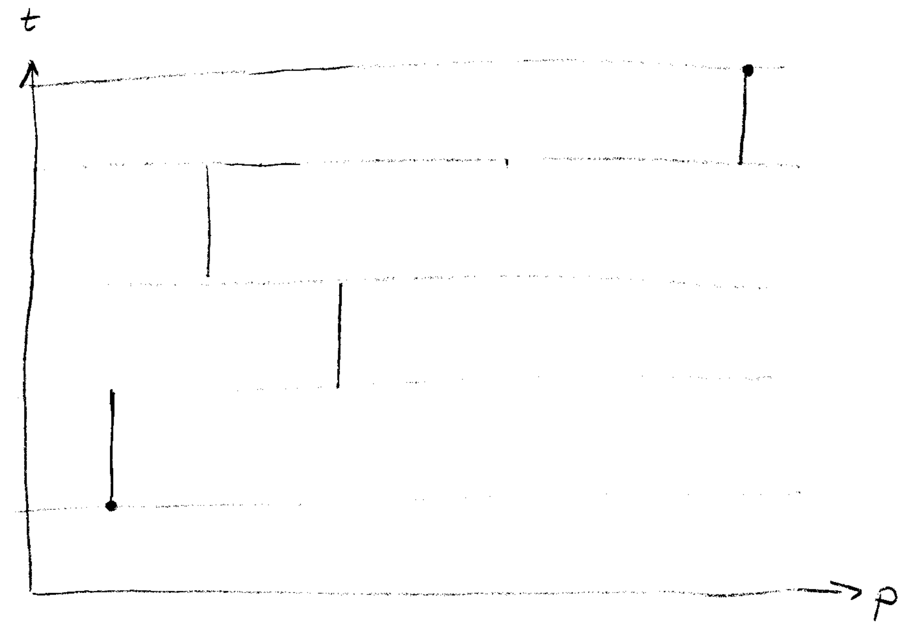


PATH INTEGRAL $= \lim_{\Delta t \to 0} \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \cdots$

MOMENTUM SPACE



TRAJECTORY IN MOMENTUM SPACE



PHASE SPACE PATH INTEGRAL

⇒ MORE GENERAL! WORKS EVEN WHEN p IS NOT QUADRATICALLY SEPARABLE IN $\mathcal{H}$

$$U(x_2, t_2;\ x_1, t_1) = \int \mathcal{D}[p(t)]\ \mathcal{D}[x(t)]$$

$$\exp\left\{ \frac{i}{\hbar} \int_{t_1}^{t_2} [p\dot{x} - \mathcal{H}(x,p)]\, dt' \right\}$$

$$\qquad\qquad 2T \qquad -(T+V)$$

When the Hamiltonian is quadratic in the momentum, and the momentum is separable, we can do the $\mathcal{D}[p(t)]$ integration to obtain the COORDINATE SPACE PATH INTEGRAL!

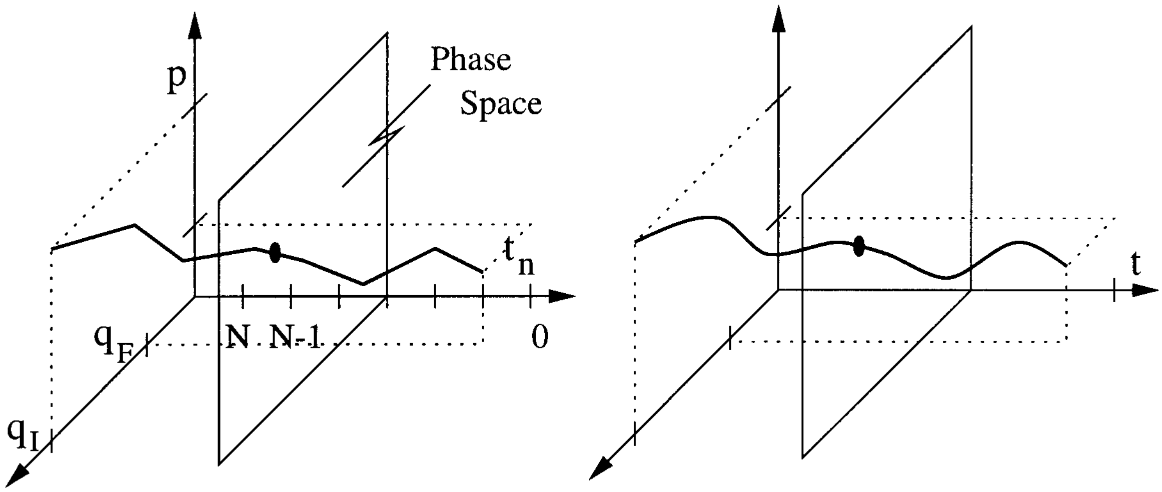
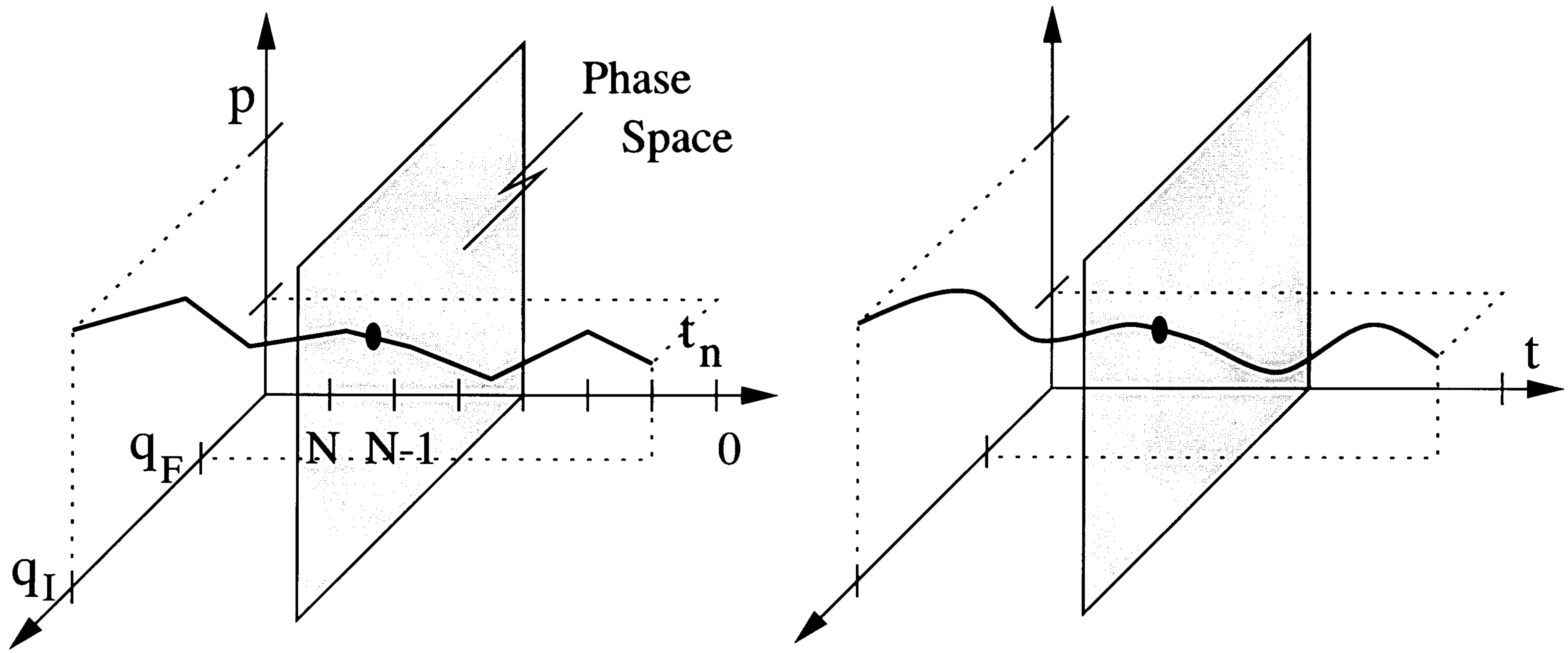


Figure 3.1: Left: visualisation of a set of phase space points contributing to the discrete time configuration integral (3.5). Right: in the continuum limit, the set of points becomes a smooth curve.



Phase
Space
p
t_n
q_F
N
N-1
0
q_I
t

PHASE SPACE PATH INTEGRAL

$$\int e^{iS/\hbar} \, D[p(t)] \, D[x(t)]$$

L [redacted] $= p\dot{x} - H(x, p)$

$\downarrow$

$$2T - (T + V)$$

$$T - V$$

$$S = \int_{t_1}^{t_2} L \, dt$$

WHEN $H \propto p^2$ AND x AND p SEPARATE

DO $\int D[p(t)] \Rightarrow$ position space path integral

QUANTUM GRAVITY $\int D[g_{\mu\nu}] \, D[\text{else}] \, e^{iS_M/\hbar}$

(annotations: $e^{iS_G/\hbar}$ with arrow; "everything" above "else")